Multifunctional variational method for description of evolution and dynamics of dissipative structures in nonequilibrium systems

V. V. Osipov
Russian Open University, Research, Development, and Production Center "Orion," Plekhanov Str. 2/46, Moscow 111123, Russia
(Received 7 May 1992)

A variational principle based on the introduction of a vector functional, each component whereof has its extremum with respect to variation of only one (or a group of) macroscopic parameter(s) of a system, is presented. The suggested method greatly facilitates the description of the characteristic parameters of stationary and time-dependent complex inhomogeneous macroscopic states occurring in nonequilibrium systems. The fruitfulness and simplicity of the presented method are illustrated by analysis of bifurcation types and by studying the shape, stability, and evolution of spike strata and autosolitions. A nonlinear theory of pulsating spike strata and autosolitons of large amplitude is developed. It is shown that the variations of the characteristic parameters of pulsating strata and autosolitons are relaxational spike auto-oscillations which may be of periodical as well as of apparently chaotic character. A simple method of analysis of quasiharmonic states is developed and, for a concrete model, it is shown that supercritical-solution bifurcations take place only if parameters of the system meet very rigorous requirements, i.e., it is confirmed that instability of the system's homogeneous states leads, as a rule, to abrupt formation of large-amplitude structures. The proposed variational principle is shown to be useful for deriving a set of ordinary differential equations, which describe in a simple way the interaction between autosolitons and turbulence in active distributed media.

PACS number(s): 05.70.Ln

I. INTRODUCTION

In recent years the interest in striking and amazing phenomena of nonlinear physics has been progressively increasing. Such phenomena are, for example, the formation of inhomogeneous macroscopic states in the form of solitons and autosolitons in physical, chemical, and biological systems. The properties of such macrostructures can be described by several macroscopic variables of diverse origin θ_i such as wave amplitude, electromagnetic-field magnitude, densities, and temperature of various particles, etc. Macroscopic inhomogeneous static states can occur in nonequilibrium as well as in equilibrium systems.

As follows from the principles of thermodynamics [1], for any thermodynamically equilibrium system there is a general functional $\mathcal{F}$ (usually called free energy) which has an extremum with respect to variation of each macroscopic variable θ_i, i.e.,

$$\frac{\delta \mathcal{F}}{\delta \theta_i}=0 . \tag{1}$$

Conditions (1) determine the equations describing thermodynamically equilibrium states, including inhomogeneous ones. Such an approach is used, for example, for the study of inhomogeneous macroscopic states occurring near various phase transition points [1].

The existence of such a general functional satisfying condition (1) is the essential property of thermodynamically equilibrium systems in which they vitally differ from nonequilibrium ones. Besides, inhomogeneous states in nonequilibrium systems may be not only static but also periodically and chaotically time dependent, i.e., pulsating structures and turbulence.

Actually, there is no ground to assert [2] that in the general case for a nonequilibrium dissipative systems there exists a single functional which has an extremum with respect to variation of any macroscopic parameter of the system. In other words, there is no general scalar functional which makes it possible to obtain the equations of stationary states from conditions (1), and the equations of time-dependent states from the following conditions:

$$\tau_i \dot{\theta}_i = -\frac{\delta \mathcal{F}}{\delta \theta_i} , \tag{2}$$

where $\dot{\theta}_i = \delta\partial_i/\partial t$ and $\delta\mathcal{F}/\delta\theta_i$ is a variational derivative of the functional $\mathcal{F}$ with respect to the variable θ_i.

In most cases the set of fundamental equations describing the properties of nonequilibrium systems is known. However, attempts to study analytically even the stationary inhomogeneous solutions of such a set of nonlinear partial differential equations have not been a success even for the case of the two most simple nonlinear equations. Time-dependent solutions are studied on the basis of numerical analysis. At the same time information about the exact spatial distribution of macroscopic variables is usually redundant. As a rule we are only interested in the characteristic parameters of these distributions (the amplitude and the width of typical areas, stability, propagation speed, auto-oscillation frequency, etc.) and their dependence on the excitation level of the system. That is why an opposite problem as to how to formulate a variational principle, on the basis of fundamental equations, which facilitates the study of evolution and dynamics of complex dissipative structures is vital.

In other words, it is promising to use the advantages of variational methods, as they make it possible to change over from studying a system of complicated nonlinear partial differential equations to the analysis of algebraic or ordinary differential equations describing characteristic parameters of complex inhomogeneous states, including time-dependent ones. This task is especially important for the correct derivation of simplified equations describing turbulence in various dissipative systems, or to be more precise, time-space stochastic auto-oscillations of an arbitrary large amplitude.

The stated problem is considered in the present paper. A variational principle is developed on the basis of three basic concepts. The main one is that for a nonequilibrium system one should not seek a single scalar functional but a vector functional in the form of a collection of some functionals, each having an extremum with respect to variation of only one (or a group of) macroscopic variable of the system, more exactly, with respect to either one or several parameters on which the variables depend.

The second concept is dealing with the fact that the method under consideration, as well as any other variational method [3], may be effective only when the qualitative form for distribution of variables is known. For most dissipative systems this condition may be considered fulfilled, because the form of such distribution, as a rule, may be determined from the general qualitative theory of dissipative structures, which has been recently developed [4–6], or from numerical analysis.

Finally, the third concept, which considerably facilitates the employment of this method, is that, according to the general theory of dissipative structures [4–6], the distributions of various macroscopic variables θ_i differ essentially from one another in form and may be described by independent coefficients. One of the advantages of variational methods is that the values of these coefficients are weakly dependent on concrete type of trial functions, which are used to describe concrete dissipative structures. Strictly speaking, this question requires studying for every particular case.

This point, and the utility of the present method for the description of evolution and dynamics of dissipative structures, is illustrated by an example of a concrete model. This model is chosen because it is the only model so far for which analytical solution, approximately describing complicated dissipative structures, has been found.

II. VARIATIONAL METHOD FOR STATIONARY INHOMOGENEOUS STATES

For clarity let us begin with the application of the variational method to the analysis of stationary inhomogeneous states occurring in dissipative media, whose properties are described by a system of differential equations of the diffusion type:

$$\tau_i \dot{\theta}_i = L_i^2 \Delta\theta_i - Q_i(\theta_1, \ldots, \theta_i, \ldots, \theta_N, A_p) \ . \tag{3}$$

In Sec. III this method is generalized for the case of equations of arbitrary type, as well as for time-dependent inhomogeneous states. In the set of Eqs. (3) the subscript $i = 1, 2, \ldots, N$, where N is the number of the macroscopic variables θ_i describing inhomogeneous states of a given dissipative system; τ_i and L_i are the characteristic time of relaxation and the length of changing of variables θ_i; A_p are the system's parameters, including those determining the excitation level.

Observe the fundamental character of the set of Eqs. (3). As a matter of fact, Eqs. (3) are equations of physical and chemical kinetics, and they are basic when inhomogeneous macroscopic states, the so-called dissipative structures, are investigated in a broad class of nonequilibrium physical, chemical, and biological systems [2,4–7].

Let us introduce a vector functional, whose ith component is

$$\Phi_i = \int [\tfrac{1}{2} L_i^2 (\nabla\theta_i)^2 + U_i(\theta_1, \ldots, \theta_i, \ldots, \theta_N, A_p) - U_0] dV \ , \tag{4}$$

where

$$U_i = \int^{\theta_i} Q(\theta_1, \ldots, \theta_i, \ldots, \theta_N, A_p) d\theta_i \tag{5}$$

and U_0 is an arbitrary constant independent of θ_i. Integration in Eq. (5) is performed at fixed values of all θ_j with $j \neq i$.

As follows from Eqs. (4) and (5), Eqs. (3) can be written as

$$\tau_i \dot{\theta}_i = -\frac{\delta\Phi_i}{\delta\theta_i} \ . \tag{6}$$

This is valid at neutral boundary conditions, i.e., when $\nabla\theta_i|_S = 0$ at the system's boundaries.

Emphasize that Eq. (6) differs essentially from Eq. (2) by the fact that a separate functional Φ_i built on the basis of the correspondence between Eqs. (6) and (3) is used for each macroscopic parameter θ_i, instead of a single functional $\mathcal{F}$. Naturally, for some groups of variables θ_i the functional Φ_i may coincide.

As follows from Eq. (6) stationary states correspond to an extremum of each of the functionals Φ_i, but with respect to variation of only one system's variable θ_i. This property of the functionals Φ_i makes it possible to use the well-known variational methods [3].

Let us formulate the variational method which best suits the analytical study of complicated dissipative structures. Note that the shape of the distributions $\theta_i(\mathbf{r})$, describing such structures, for most systems can be determined easily from the general qualitative theory of large-amplitude dissipative structures [4–6]. Therefore, we can approximately describe the distributions $\theta_i(\mathbf{r})$ as

$$\theta_i(\mathbf{r}) = \sum_{j=1}^{n_i} \theta_j^{(i)}(\mathbf{r}, C_{j1}^{(i)}, C_{j2}^{(i)}, \ldots, C_{jk}^{(i)}, \ldots, C_{jm_j}^{(i)}) \ , \tag{7}$$

where $\theta_j^{(i)}$ are trial analytical functions which best describe the qualitative shape of distributions $\theta_i(\mathbf{r})$; $C_{jk}^{(i)}$ are the sought-for coefficients; $j = 1, 2, \ldots, n_i$; n_i is the number of trial functions; $k = 1, 2, \ldots, m_j$; m_j is the number of the coefficients on which each trial function depends. Function (7) must satisfy the given boundary conditions at any value of $C_{jk}^{(i)}$.

The substitution of (7) into Eq. (4) yields a set of functionals Φ_i which are naturally independent of the coordinates, but depend on the coefficients $C_{jk}^{(i)}$. In the stationary case from condition (6) it follows that each of the functionals Φ_i has an extremum with respect to an arbitrary variation of only the variable θ_i; hence for all i, j, and k we have

$$\frac{\partial \Phi_i}{\partial C_{jk}^{(i)}}=0 . \tag{8}$$

Equations (8) forms a set of algebraic equations in coefficients C_{ij} determining the function $\theta_i(\mathbf{r})$ of (7). As Φ_i depends on A_p, C_{ij} are functions of the parameters A_p including a parameter characterizing the excitation level of the system. Functions $C_{ij}(A_p)$ determine the evolution of dissipative structures when the excitation level A_p and other parameters of the system vary (Secs. IV and V).

III. VARIATIONAL METHOD FOR DESCRIPTION OF INHOMOGENEOUS STATES VARYING IN TIME

Equations describing macroscopic properties of nonequilibrium dissipative structures can be written for a rather general case in the form

$$\tau_i \dot{\theta}_i=\hat{F}_i(\theta_1,\dots,\theta_i,\dots,\theta_N,A_p) , \tag{9}$$

where $\hat{F}_i$ is a certain functional; τ_i is the variable θ_i relaxation time incident to the corresponding dissipative process in the system under consideration.

Following the variational method presented in Sec. II, we seek solutions θ_i in the form (7). Thus

$$\dot{\theta}_i(\mathbf{r},t)=\sum_{j=1}^{n_i}\sum_{k=1}^{m_j}\frac{\partial \theta_j^{(i)}(\mathbf{r})}{\partial C_{jk}^{(i)}}\dot{C}_{jk}^{(i)} . \tag{10}$$

Let us substitute Eq. (10) into Eq. (9). Multiplying the resultant equation by $\partial\theta_n^{(i)}/\partial C_{nl}^{(i)}$ and integrating over the space we obtain a set of ordinary differential equations in coefficients C_{ij}:

$$\tau_i\sum_{j=1}^{n_i}\sum_{k=1}^{m_j}\dot{C}_{jk}^{(i)}\int\left[\frac{\partial\theta_j^{(i)}}{\partial C_{jk}^{(i)}}\frac{\partial\theta_n^{(i)}}{\partial C_{nl}^{(i)}}\right]dV=\int\left[\hat{F}_i\frac{\partial\theta_n^{(i)}}{\partial C_{nl}^{(i)}}\right]dV . \tag{11}$$

Recall that $i=1,2,\dots,N$, where N is the number of macroscopic variables θ_i describing an inhomogeneous state of the given nonequilibrium system; $k=1,2,\dots,m_i$ and m_j is the number of the coefficients $C_{jk}^{(i)}$ which the trial function $\theta_j^{(i)}$ depends on; $j=1,2,\dots,n_j$, where n_j is the number of trial functions $\theta_j^{(i)}(\mathbf{r})$ approximating $\theta_i(\mathbf{r},t)$. Note that when the form of distributions $\theta_i(\mathbf{r})$ is time dependent, the numbers n_i and m_i in Eq. (10), which describe $\theta_i(\mathbf{r},t)$, may differ from n_i and m_i in Eq. (7), which determine the form of stationary distributions $\theta_i(\mathbf{r})$.

The general equation (11) makes it possible to derive a set of ordinary differential equations describing the evolution and dynamics of the characteristic parameters of inhomogeneous states in dissipative systems on the basis of fundamental partial differential equations (9).

In the case of Eq. (3)

$$\hat{F}_i=L_i^2\Delta\theta_i-Q_i(\theta_1,\dots,\theta_i,\dots,\theta_N,A_p) .$$

Substituting such $\hat{F}_i$ into Eq. (11) and taking into account the definition (5), after obvious transformations under neutral boundary conditions we obtain

$$\tau_i\sum_{j=1}^{n_i}\sum_{k=1}^{m_j}\dot{C}_{jk}^{(i)}\int\left[\frac{\partial\theta_j^{(i)}}{\partial C_{jk}^{(i)}}\frac{\partial\theta_j^{(i)}}{\partial C_{nl}^{(i)}}\right]dV =\frac{\partial}{\partial C_{nl}^{(i)}}\int\left[\frac{L_i^2(\nabla\theta_i)^2}{2} +U_i(\theta_1,\dots,\theta_i,\dots,\theta_N,A_p)-U_0\right]dV . \tag{12}$$

In the stationary case, i.e., when $\dot{C}_{jk}^{(i)}=0$, condition (8) follow from Eq. (12) and expression (4).

Note that Eq. (11) can also be derived on the basis of the Gauss variational method [3]. Indeed, following the concept of this method we can regard θ_i and $\dot{\theta}_i$ as independent variables and introduce a set of dissipative functionals

$$D_i=\tfrac{1}{2}\int[\tau_i\dot{\theta}_i-F_i]^2dV . \tag{13}$$

As follows from Eqs. (9) and (13)

$$\frac{\partial D_i}{\partial\dot{\theta}_i}=\tau_i\dot{\theta}_i-F_i=0 . \tag{14}$$

Thus each of the dissipative functionals D_i has an extremum with respect to an arbitrary variation of a variable $\dot{\theta}_i$. From this property of functionals (13) it follows that

$$\frac{\partial}{\partial\dot{C}_{ik}}\tfrac{1}{2}\int[\tau_i\dot{\theta}_i-F_i]^2dV=0 . \tag{15}$$

The substitution of Eq. (10) into Eq. (15) and subsequent differentiating yields Eq. (11).

Equation (10) and specifically Eq. (12) greatly facilitate studying the stability of inhomogeneous states, the description of their transformation dynamics, and the analysis of periodically and stochastically oscillating dissipative structures, i.e., turbulence in concrete systems. Some examples of these effects are considered in the following sections where spike strata and autosolitons, which can occur in various nonequilibrium physical, chemical, and biological systems [4–6], are analyzed.

IV. EVOLUTION OF SPIKE STATIC STRATA

Autosolitons and strata are localized highly nonequilibrium regions occurring in various nonequilibrium systems including electron-hole and gas plasma, heated gas mixture, ferroelectric photoconductors, composite superconductors, as well as in semiconductor, magnetic, and optical nonlinear media [4–6]. Properties of many of these systems can be described by a set of two diffusion equations of type (3) [4,6].

Evolution of static strata and autosolitons in these different physical media can be fairly easily analyzed on the basis of the variational method developed in Sec. II because the qualitative shape of inhomogeneous states in a concrete physical system can be found from the general qualitative theory of autosolitons and strata [4–6].

However, because of the complexity of nonlinear equations describing strata in real physical systems, even approximate analytical solutions determining the properties of autosolitons and strata have not been found. The only exception here is the classical biological Gierer-Meinhardt model [3,8,9]:

$$\tau_\theta \dot{\theta} = l^2 \Delta\theta + A\theta^2\eta^{-1} + 1 - \theta \; , \tag{16}$$

$$\tau_\eta \dot{\eta} = L^2 \Delta\eta - \eta + \theta^2 \; , \tag{17}$$

where $\theta \equiv \theta_1$, $\eta \equiv \theta_2$. For this model approximate analytical solutions, which describe the distributions $\theta(x)$ and $\eta(x)$ in the form of a stratum or a one-dimensional autosoliton at $\epsilon \equiv l/L \ll 1$ and $\epsilon \ll A^2 < 1$, have been found [4,10]. So it is natural to choose model (16),(17), as a test for verifying the accuracy of the results obtained by the presented variational method.

In accordance with the general qualitative theory [4,6] at $\epsilon = l/L \ll 1$ model (16),(17) allow solutions in the form of a narrow spike stratum. Such a stratum is an inhomogeneous state in which the shape of distribution $\theta(x)$ is a large-amplitude spike with the size of order l, and the distribution $\eta(x)$ changes smoothly with the characteristic length of order L [Fig. 1(a)].

In a simple case, when size $\mathcal{L}$ of the sample satisfies the condition $l \ll \mathcal{L} \ll L$ and at its boundaries $\theta'_x = \eta'_x = 0$ [at $x = \pm\mathcal{L}/2$ in Fig. 1(a)], we can assume $\eta(x) \cong \text{const}$ and seek the solution in the form

$$\theta(x) = C_{11} \cosh^{-n}\left[\frac{\xi x}{l}\right] + C_{12}, \quad \eta = C_{21} \; . \tag{18}$$

According to Eq. (4) let us introduce the functionals

$$\Phi_1 = \int_0^{\mathcal{L}/2} [\tfrac{1}{2} l^2 (\theta'_x)^2 + U_1(\theta(x), \eta(x))] dx \tag{19}$$

and

$$\Phi_2 = \int_0^{\mathcal{L}/2} [\tfrac{1}{2} L^2 (\eta'_x)^2 + U_2(\theta(x), \eta(x))] dx \; , \tag{20}$$

where

$$U_1 = -\frac{A\theta^3\eta^{-1}}{3} - \theta + \frac{\theta^2}{2} \; , \tag{21}$$

$$U_2 = -\theta^2\eta + \frac{\eta^2}{2} \; . \tag{22}$$

It is easy to see that the substitution of Eqs. (19)–(22) into Eq. (6) at neutral boundary conditions yields Eqs. (16) and (17). Thus, for a stationary case functionals Φ_1 and Φ_2 have extrema with respect to variation of the variables $\theta = \theta_1$ and $\eta = \theta_2$, respectively. In other words, Φ_1 and Φ_2 satisfy condition (8), which, for the problem under consideration, can be written as

$$\frac{\partial \Phi_1}{\partial C_{11}} = 0, \quad \frac{\partial \Phi_1}{\partial C_{12}} = 0, \quad \frac{\partial \Phi_2}{\partial C_{21}} = 0 \; . \tag{23}$$

For definiteness assume that $n = 2$ and $\xi = \frac{1}{2}$ in (18). Then the substitution of Eq. (18) into Eqs. (19)–(22) yields

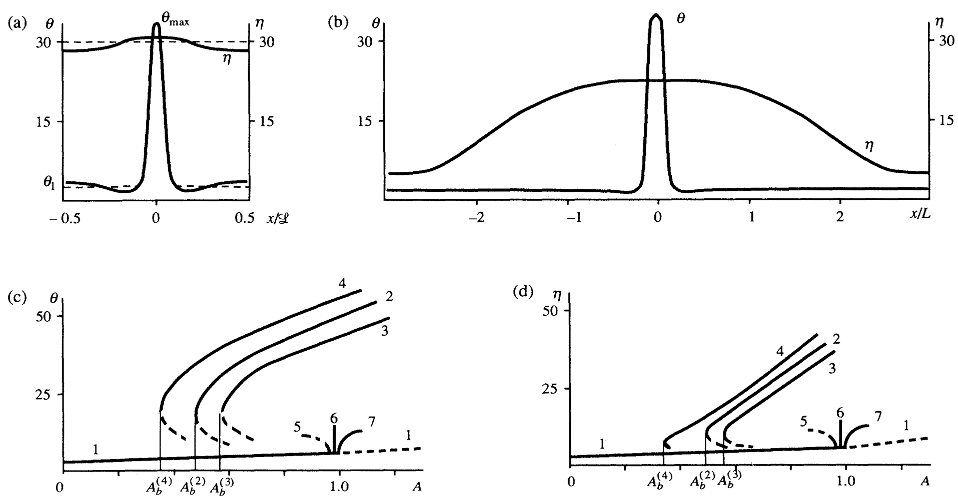

FIG. 1. Evolution of a stratum and an autosoliton under change of the excitation level (parameter A) of a system. (a) and (b) The distributions θ and η are shown in the form of a stratum and an autosoliton, respectively [the numerical calculation results of the models (16) and (17) at $A = 0.9$ and $\epsilon = 10^{-2}$ (Ref. [6]); (c) and (d) bifurcation characteristics, i.e., dependences of the values of θ and η on the parameter A in the center of a stratum or an autosoliton, are shown (curves 1 indicate dependences $\theta = \theta_h$ and $\eta = \eta_h$ for an homogeneous state; curves 2 for a stratum; curves 3 for an autosoliton at $n = 2$ and $\xi = \frac{1}{2}$; curves 4 for an autosoliton at $n = 1$ and $\xi = 1$; curves 5–7 for low-amplitude periodic strata at different values of ϵ; curve 5, $\epsilon < \epsilon_c$; curve 6, $\epsilon = \epsilon_c$; curve 7, $\epsilon_c < \epsilon < \epsilon_m$). Solid lines show distributions $\theta(x)$ and $\eta(x)$ at $L \gg \mathcal{L}$, dashed lines illustrate those distributions at $L > \mathcal{L}$ in (a). Dashed lines correspond to the unstable states in (c) and (d).

$$\Phi_1=\tfrac{4}{5}lC_{11}^2-2lAC_{11}C_{12}^2C_{21}^{-1}-\tfrac{4}{3}lAC_{11}^2C_{12}C_{21}^{-1}-\tfrac{16}{45}lAC_{11}^3C_{21}^{-1}-2lC_{11}+2lC_{11}C_{12}-\tfrac{1}{6}A\mathcal{L}C_{12}^3C_{21}^{-1}-\tfrac{1}{2}\mathcal{L}C_{12}+\tfrac{1}{4}\mathcal{L}C_{12}^2\ , \tag{24}$$

$$\Phi_2=\frac{\mathcal{L}}{4}C_{21}^2-\frac{\mathcal{L}}{2}C_{21}\left[\frac{8l}{3\mathcal{L}}C_{11}^2+\frac{8l}{\mathcal{L}}C_{11}C_{12}+C_{12}^2\right]. \tag{25}$$

From Eqs. (24) and (25) we find

$$\frac{\partial\Phi_1}{\partial C_{11}}=\tfrac{8}{5}lC_{11}-2lAC_{12}^2C_{21}^{-1}-\tfrac{8}{3}lAC_{11}C_{12}C_{21}^{-1}-\tfrac{16}{15}lAC_{11}^2C_{21}^{-1}-2l+2lC_{12}\ , \tag{26}$$

$$\frac{\partial\Phi_1}{\partial C_{12}}=-4lAC_{11}C_{12}C_{21}^{-1}-\tfrac{4}{3}lAC_{11}^2C_{21}^{-1}-\tfrac{1}{2}A\mathcal{L}C_{12}^2C_{21}^{-1}-\tfrac{1}{2}\mathcal{L}+\tfrac{1}{2}\mathcal{L}C_{12}+2lC_{11}\ , \tag{27}$$

$$\frac{\partial\Phi_2}{\partial C_{21}}=\frac{\mathcal{L}}{2}\left[C_{21}-\frac{8l}{3\mathcal{L}}C_{11}^2-\frac{8l}{\mathcal{L}}C_{11}C_{12}-C_{12}^2\right]. \tag{28}$$

Equations (26)–(28), together with Eq. (23), form a set of algebraic equations in the coefficients $C_{11}^{(0)}$, $C_{12}^{(0)}$, and $C_{21}^{(0)}$ describing characteristic parameters of a static stratum. The approximate solution of this set of equations at $\mathcal{L}\gg l$ and $C_{11}\gg C_{12}$ is

$$C_{21}^{(0)}=\eta^{(0)}=\frac{A^2\mathcal{L}}{12l}\left[1\pm\left[1-\frac{A_b^2}{A^2}\right]^{1/2}\right], \tag{29a}$$

where $A_b=(24l/\mathcal{L})^{1/2}$, and

$$C_{11}^{(0)}\equiv\theta_{\max}-\theta_1=\frac{3C_{21}^{(0)}}{2A}, \tag{29b}$$

$$C_{12}^{(0)}\equiv\theta_1=\frac{\eta^{(0)}}{2A}\left[1-\sqrt{1-4A/\eta^{(0)}}\right]. \tag{29c}$$

The substitution of (29) into (18) yields the sought-for distribution $\theta(x)$.

Notice that the found characteristic parameters (29) of distribution $\theta(x)$ obtained on the basis of variational method presented in Sec. II are in perfect agreement with their true values reported in [4,6]. True distribution $\theta(x)$ practically coincides with Eqs. (29) and (18) at $n=2$ and $\xi=\frac{1}{2}$ because the exact value $\xi=\frac{1}{2}(1-4A/\eta^{(0)})^{1/2}\cong\frac{1}{2}$ at $\mathcal{L}\gg l$ and $A>Ab$ [4,6].

For a more precise analysis of the problem we shall regard quantities n and ξ in Eq. (18) as independent coefficients ($n\equiv C_{13}$ and $\xi\equiv C_{14}$). Then we arrive at a set of five algebraic equations in the coefficients $C_{11}^{(0)}$, $C_{12}^{(0)}$, $C_{13}^{(0)}$, $C_{14}^{(0)}$, and $C_{21}^{(0)}$. This set of equations satisfies the exact solution found in Ref. [10] on the basis of the analytical study of the considered problem. This exact solution coincides with Eqs. (29) and (18) for $n=2$ and $\xi=\frac{1}{2}(1-4A/\eta^{(0)})^{1/2}$ [4,6].

We can also note that Eqs. (29) relatively weakly depend on the values of n and $\xi=n^{-1}$. Equations (29) also do not alter considerably if we use $\exp(-x^2/l^2)$ instead of $\cosh^{-n}(\xi x/l)$ in Eq. (18). In other words, the main results obtained by the variational method presented in Sec. II relatively weakly depend on the accuracy of the description of distribution $\theta(x)$. This conclusion confirms one of the basic advantages of variational methods (see also Sec. VI).

It can be seen from Eqs. (29) and (18) that the amplitude $\theta_{\max}\cong C_{11}^{(0)}$ of a stratum decreases as A decreases. At the point $A=A_b$, where $d\eta^{(0)}/dA=\infty$ and $d\theta_{\max}/dA=\infty$ [Figs. 1(c) and 1(d)], the stratum with the large amplitude

$$\theta_{\max}=\left[\frac{3\mathcal{L}}{8l}\right]^{1/2}\gg 1 \tag{30}$$

disappears abruptly. Equations (29) and (18) also describe the stratum evolution when the size of the system varies. Specifically, it follows from them that the stratum disappears abruptly when the system's size becomes less than the quantity $\mathcal{L}_c=24lA^{-2}$.

V. STABILITY OF STATIC AND AUTO-OSCILLATIONS OF PULSATING SPIKE STRATA

As is seen from the analysis of the formation conditions as well as from the numerical study of a spike stratum, the shape of this stratum is unchanged in the process of auto-oscillations and qualitatively coincides with the shape of a static stratum [4,6]. In other words, the process of pulsating of a spike stratum is auto-oscillation of the stratum amplitude $\theta_{\max}-\theta_1\equiv C_{11}$ and its characteristic values $\theta_1\equiv C_{12}$ and $\eta^{(0)}\equiv C_{21}$. That is why expressions (18) at $n=2$, $\xi=\frac{1}{2}$, and time-dependent coefficients C_{11},C_{12},C_{21} may be used for the study of pulsating strata dynamics in model (16),(17).

The substitution of Eqs. (18)–(22) into (12) and calculation of the corresponding integrals yields

$$\tfrac{2}{3}\tau_\theta\dot{C}_{11}+\tau_\theta\dot{C}_{12}=-\frac{1}{2l}\frac{\partial\Phi_1}{\partial C_{11}}\ , \tag{31a}$$

$$\frac{4l}{\mathcal{L}}\tau_\theta\dot{C}_{11}+\tau_\theta\dot{C}_{12}=-\frac{2}{\mathcal{L}}\frac{\partial\Phi_1}{\partial C_{12}}\ , \tag{31b}$$

$$\tau_\eta\dot{C}_{21}=-\frac{2}{\mathcal{L}}\frac{\partial\Phi_2}{\partial C_{21}}\ . \tag{31c}$$

Substituting Eqs. (26)–(28) into Eqs. (31) and taking into account $\mathcal{L}\gg l$ and $C_{11}\gg C_{12}$, after obvious rearrangement we obtain equations describing the time-dependent characteristic parameters of the stratum

$$\tfrac{2}{3}\alpha\dot{C}_{11}=\frac{8A}{15}C_{11}^2C_{21}^{-1}+\tfrac{4}{3}AC_{11}C_{12}C_{21}^{-1}-\frac{4}{5}\left[1+\frac{11l}{\mathcal{L}}\right]C_{11}\ , \tag{32}$$

$$\alpha\dot{C}_{12}=AC_{12}^2C_{21}^{-1}-C_{12}+1-\frac{8lA}{15\mathcal{L}}\left[1-\frac{6l}{\mathcal{L}}\right]^{-1}C_{11}^2C_{21}^{-1}\ , \tag{33}$$

$$\dot{C}_{21}=\frac{8l}{3\mathcal{L}}C_{11}^2+\frac{8l}{\mathcal{L}}C_{11}C_{12}+C_{12}^2-C_{21}\ , \tag{34}$$

where $\alpha \equiv \tau_\theta/\tau_\eta$ and time is measured in the units of τ_η.

To study stability of a static stratum we seek the solution of Eqs. (32)–(34) in the form

$$C_{ij}(t)=C_{ij}^{(0)}+\delta C_{ij}\exp(-\gamma t) , \quad (35)$$

where $C_{ij}^{(0)}$ is determined by Eqs. (29). Linearizing Eqs. (32)–(34) with respect to small perturbations $\delta C_{ij} \ll C_{ij}^{(0)}$ we find an equation determining the quantity γ:

$$\gamma^3 - a\gamma^2 + b\gamma - c = 0 , \quad (36)$$

where

$$a=\alpha^{-1}(\tfrac{1}{5}-\alpha)+\alpha^{-1}\epsilon\left[\frac{12}{AM}(1+\alpha)-\frac{6}{5}\right],$$

$$b\cong\alpha^{-1}[1+\tfrac{6}{5}(M-1-\alpha^{-1})] +\epsilon\alpha^{-1}\left[6\alpha+\frac{72}{A}+\tfrac{6}{5}(1+\alpha^{-1}) -12(\alpha AM)^{-1}-24(AM)^{-1}\right],$$

$$c=\alpha^{-2}\left[\tfrac{6}{5}(1-M)+\epsilon\left[\frac{12}{A}(1-A)-\tfrac{9}{5} +\frac{72(AM+1)(5-M)}{25AM}\right]\right],$$

$$M=\left[1\pm\left[1-\frac{A_b^2}{A^2}\right]^{1/2}\right].$$

Analysis of these equations shows that conclusions which can be drawn from it are close to those drawn from the equation

$$\tfrac{2}{3}\alpha C_{21}^{(0)}\gamma^2+(\tfrac{4}{5}C_{21}^{(0)}+\tfrac{4}{3}A-\tfrac{2}{3}\alpha C_{21}^{(0)}) +\left[\frac{24}{5A^2}\frac{l}{\mathcal{L}}(C_{11}^{(0)})^2-\tfrac{4}{5}C_{21}^{(0)}\right]=0 . \quad (37)$$

Equations (37) follows from linearized Eqs. (32)–(34) if we take into account that at $\mathcal{L} \gg l$ and $A > A_b$ according to Eq. (29) the quantity $C_{12}^{(0)} \cong 1$ and much less than $C_{21}^{(0)}$ and $C_{11}^{(0)}$, and according to Eqs. (33) and (29) the amplitude of oscillations of the quantity C_{12}, i.e., δC_{12}, is much less than δC_{11}, δC_{21}.

The principal result following from Eqs. (36) and (37) is that at $C_{21}^{(0)} < A^2\mathcal{L}(6l)^{-1}$ for one of the values of γ, the quantity $\mathrm{Re}(\gamma)<0$ regardless of α. In other words, the smaller-amplitude stratum, with parameters corresponding to the minus in Eqs. (29), is unstable regardless of $\alpha \equiv \tau_\theta/\tau_\eta$. This conclusion confirms the general result of the theory of strata [4,6], according to which the lower branch of the curves $\eta^{(0)}(A)$ and $\theta_{\max}(A)$ indicated by the dashed lines in Figs. 1(c) and 1(d) corresponds to unstable states in the form of the smaller amplitude stratum.

It follows from Eqs. (36) and (37) together with Eqs. (29) that a large-amplitude stratum is unstable only when a condition, which can be approximately written as

$$\alpha \equiv \tau_\theta/\tau_\eta < 1.2+\frac{24l}{\mathcal{L}A}\{1+[1-(A_b/A)^2]^{1/2}\}^{-1} , \quad (38)$$

is satisfied. From Eq. (38) we can see that at α less than a quantity of order one and $\mathcal{L} \gg l$ the stratum is unstable practically regardless of the value of A. This conclusion is consistent with the result of the general theory of spike strata stability [4,6].

On the other hand, a stratum is stable if an inequality opposite to (38) is valid. Substituting the boundary value of $A = A_b$ from Eq. (29) into this inequality we obtain that when

$$\alpha > 1.2+\left[\frac{24l}{\mathcal{L}}\right]^{1/2} , \quad (39)$$

the larger-amplitude stratum [corresponding to the plus in Eq. (29)] is stable in the whole range of its existence up to the point $A = A_b$ [Figs. 1(c) and 1(d)] where the stratum abruptly disappears.

The linear theory of static spike strata stability [4,6] predicts that pulsating spike strata can occur in a system at $\alpha \lesssim 1$. On the basis of the presented variational method we have obtained a set ordinary differential Eqs. (32)–(34) describing nonlinear auto-oscillations of the characteristic parameters of pulsating strata, including arbitrary large-amplitude ones.

It follows from the results of numerical analysis of Eqs. (32)–(34) that the amplitude of auto-oscillations of the quantity $C_{12} \equiv \theta_1$ is small in comparison with the amplitudes of auto-oscillations of the quantities $C_{11} \equiv \theta_{\max} - \theta_1$ and $C_{21} \equiv \eta^{(0)}$ (Fig. 2). This result was used for deriving Eq. (37) and, as it might seem, it enables us to restrict our consideration of a pulsating stratum to investigate a set of two equations (32) and (34) putting $C_{12}=C_{12}^{(0)}=1$ [as follows from Eq. (29)].

However, a set of two nonlinear differential equations allows solutions only in the form of either damping, increasing or strictly periodical auto-oscillations, depending on the parameters of the system. At the same time analysis of the set of three equations (32)–(34) shows that undamped auto-oscillations of main parameters of a pulsating stratum may not be strictly periodical and at some values of the system's parameters they have essentially stochastic character (Fig. 2). In other words, here we observe the well-known result of the general theory of auto-oscillations [11,12]: a transition from a set of two nonlinear differential equations to a set of three or more ones can be accompanied by arising stochastic auto-oscillations.

Analysis of Eqs. (32)–(34) also shows that undamped auto-oscillations of stratum amplitude can occur in relatively narrow ranges of the values of α and A. Moreover, the initial values of coefficients C_{11}, C_{12}, and C_{21} must be chosen relatively close to the values of $C_{11}^{(0)}$, $C_{12}^{(0)}$, $C_{21}^{(0)}$ determined from Eqs. (29). In other words, the attracting region of the state in the form of a stationary pulsating stratum with respect to initial perturbations is very small.

When $A = 0.745$ and $l/\mathcal{L} = 10^{-2}$ the homogeneous state is stable, but in a system with $\alpha > 1$ a stable static large-amplitude stratum can be excited; a pulsating stratum (Fig. 2) occurs at $0.7 < \alpha < 1$; at $\alpha < 0.7$ auto-

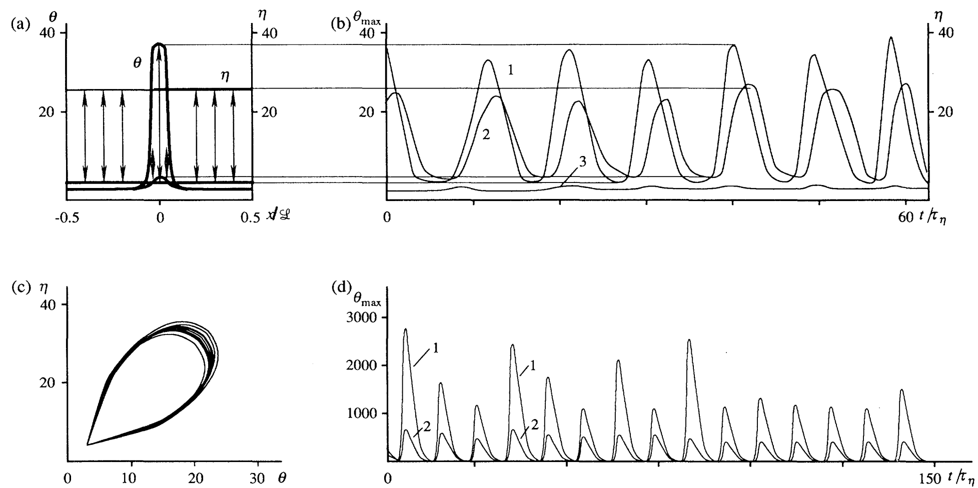

FIG. 2. (a) A pulsating spike stratum in small-size systems; (b) and (d) auto-oscillations of the stratum amplitude, i.e., of the quantity $\theta_{max} \cong C_{11}$ (curve 1), as well as of the quantities $\eta(0)=C_{21}$ (curve 2) and $\theta_1=C_{12}$ (curve 3). The results of the numerical analysis of Eqs. (32)–(34) at $l/\mathcal{L}=10^{-2}$, $\alpha=0.75$ are shown in (b) for $A=0.745$ and in (d) for $A=3$. (c) illustrates the θ-η-plane projection of the phase portrait of the auto-oscillations shown in (b).

oscillations of the stratum are damped and an homogeneous state forms in the system.

At $A=3$ the homogeneous state is unstable, but at $\alpha>1.2$ and $l/\mathcal{L}=10^{-2}$ a stable static stratum can be excited in the system; a pulsating stratum [Fig. 2(d)] occurs at $0.3<\alpha<1.2$; at $\alpha<0.3$ the pulsating stratum disappears and homogeneous relaxational spike auto-oscillations, the theory of which is developed in Ref. [13], occur in the system.

As A increases, the amplitude of auto-oscillations of the stratum, that is, the quantity $\Delta\theta_{max}$, increases sharply: at $A=0.745$ the value of $\Delta\theta_{max}\cong 30$ [Figs. 2(b) and 2(c)] and at $A=3$ the value of $\Delta\theta_{max}\cong 2800$ [Fig. 2(d)].

As A decreases, the quantity $\Delta\theta_{max}$ also decreases and at some value of A slightly greater than A_b [Figs. 1(c) and 1(d)], the pulsating large-amplitude stratum disappears abruptly and the stable homogeneous state forms in the system. It follows from the considered example that on the basis of the presented variational method the evolution of static, as well as pulsating strata, can be investigated easily.

Note that the dynamic equations (32)–(34) describe kinetics of stratum formation as well. Specifically, they allow us to find the value of perturbation, more exactly, the initial values of coefficient C_{11} in Eq. (18) at which spontaneous stratum formation begins at a given value of A.

We have considered the simplest case of a stratum occurring in a sample of the size $\mathcal{L} \ll L$ when the parameter η can be assumed independent of x. The dependence $\eta(x)$ in a sample of the size $\mathcal{L} \le L$ can be taken into account by using the following expressions instead of Eq. (18):

$$\theta(x)=C_{11}\cosh^{-2}\left[\frac{x}{2l}\right]+C_{12}+C_{13}\cos\left[\frac{2\pi x}{\mathcal{L}}\right], \quad (40)$$

$$\eta(x)=C_{21}+C_{22}\cos\left[\frac{2\pi x}{\mathcal{L}}\right]. \quad (41)$$

Analysis shows that the coefficients C_{13} and C_{22} decrease proportionally to the quantity $\mathcal{L}/L$ and the main results, including Eqs. (29) for a static stratum, as well as condition (38) and Eqs. (32)–(34) for a pulsating stratum, remain valid even for the case $\mathcal{L}=2L$. Results obtained in the following sections, where another limit case ($\mathcal{L} \gg L$) is analyzed, lead to the same conclusions.

VI. EVOLUTION OF STATIC SPIKE AUTOSOLITONS

An autosoliton occurs in an extended system of the size $\mathcal{L} \gg L$, and in the one-dimensional case it is a complex shape solitary stratum [Fig. 1(b)]. An autosoliton is a localized large-amplitude state which at the periphery asymptotically approaches a stable homogeneous state of the system $\theta=\theta_h$, $\eta=\eta_h$ [4,6]. One can easily see that the homogeneous state of model (16),(17) is

$$\theta_h=1+A, \quad \eta_h=(1+A)^2. \quad (42)$$

This state is stable at $A<A_c$, where [5,6]

$$A_c=(1+\epsilon)^2(1-2\epsilon-\epsilon^2)^{-1}. \quad (43)$$

When $\epsilon=l/L \ll 1$, the quantity $A_c=1+4\epsilon\cong 1$.

It follows from the numerical analysis and from the general qualitative theory of autosolitons [4,6] that at $\epsilon=l/L \ll 1$ the solution of model (16),(17) may have the form of a spike autosoliton [Fig. 1(b)]. Such an autosoliton is a symmetrical about the center of the solitary state [the point $x=0$ in Fig. 3(a)] in which distribution $\theta(x)$ is a spike described by the same function as in Eq. (18). Distribution $\eta(x)$ smoothly varies everywhere with the characteristic length of the order L. Far from the spike distributions $\theta(x)$ and $\eta(x)$ approach the values of θ_h and η_h, respectively, approximately following the law $\exp(-|x|/L)$. According to these results we shall seek the distributions $\theta(x)$ and $\eta(x)$ in the form

$$\theta = C_{11}\cosh^{-2}\left[\frac{x}{2l}\right] + C_{12}\exp\left[-\frac{|x|}{L}\right] + \theta_h \,, \quad (44)$$

$$\eta = C_{21}\exp\left[-\frac{|x|}{L}\right] + \eta_h \,, \quad (45)$$

where $|C_{11}| \gg |C_{12}|$, $L \gg l$, and $A < 1$. It would be more correct to use $\cosh^{-1}(x/L)$ instead of $\exp(-|x|/L)$ in Eqs. (44) and (45). But the difference, as can be shown, weakly influences the results given below.

To analyze an autosoliton we introduce the following functionals instead of (19) and (20):

$$\Phi_1 = \int_0^\infty \left[\frac{l^2(\theta'_x)^2}{2} + U_1(\theta(x),\eta(x)) - U_1(\theta_h,\eta_h)\right] dx \,, \quad (46)$$

$$\Phi_2 = \int_0^\infty \left[\frac{L^2(\eta'_x)^2}{2} + U_2(\theta(x),\eta(x)) - U_2(\theta_h,\eta_h)\right] dx \,, \quad (47)$$

where U_1 and U_2 are defined by Eqs. (21) and (22); the quantities $U_1(\theta_h,\eta_h)$ and $U_2(\theta_h,\eta_h)$ are added to functionals (46) and (47) to avoid the divergency of the integrals at the infinity.

The substitution of Eqs. (44) and (45) into Eqs. (46) and (47) yields

$$\Phi_1 = \frac{2l}{15}C_{11}^2 + \left[\frac{1}{2} - \frac{A\theta_h}{C_{21}+\eta_h}\right]\left[\frac{4l}{3}C_{11}^2 + 4lC_{11}C_{12} + \frac{L}{2}C_{12}^2\right] - \frac{A}{3(C_{21}+\eta_h)}\left[\frac{16l}{15}C_{11}^3 + 4lC_{11}^2C_{12} + 6lC_{11}C_{12}^2 + \frac{L}{3}C_{12}^3\right] + \frac{A}{C_{21}+\eta_h}\left[2lC_{11}C_{21} + \frac{L}{2}C_{12}C_{21}\right] \,, \quad (48)$$

$$\Phi_2 = \frac{L}{2}C_{21}^2 - C_{21}\left[\theta_h(C_{12}L + 4lC_{11}) + \frac{L}{2}C_{12}^2 + \frac{4l}{3}C_{11}^2\right] \,. \quad (49)$$

It will be shown below that $C_{21} \simeq \epsilon^{-1} \gg 1$ so the small terms, such as $C_{21}^{-1}\ln C_{21}$ are ignored in expression (48). Deriving Eqs. (48) and (49), we take into account $\epsilon = l/L \ll 1$ and $|C_{11}| \gg |C_{12}|$. It follows from Eqs. (48) and (49) that

$$\frac{\partial\Phi_1}{\partial C_{11}} = \frac{4l}{15}C_{11} + \left[\frac{1}{2} - \frac{A\theta_h}{C_{21}+\eta_h}\right]\left[\frac{8l}{3}C_{11} + 4lC_{12}\right] + \frac{2AlC_{21}}{C_{21}+\eta_h} - \frac{A}{3(C_{21}+\eta_h)}\left(\tfrac{16}{5}lC_{11}^2 + 8lC_{11}C_{12} + 6lC_{12}^2\right) \,, \quad (50)$$

$$\frac{\partial\Phi_1}{\partial C_{12}} = \left[\frac{1}{2} - \frac{A\theta_h}{C_{21}+\eta_h}\right](4lC_{11} + LC_{12}) + \frac{ALC_{21}}{(C_{21}+\eta_h)} - \frac{A}{3(C_{21}+\eta_h)}(4lC_{11}^2 + 12lC_{11}C_{12} + LC_{12}^2) \,, \quad (51)$$

$$\frac{\partial\Phi_2}{\partial C_{21}} = LC_{21} - \theta_h(C_{12}L + 4lC_{11}) - \frac{L}{2}C_{12}^2 - \frac{4l}{3}C_{11}^2 \,. \quad (52)$$

Equations (50)–(52) together with Eq. (23) constitute a set of algebraic equations determining characteristic parameters of a static autosoliton, i.e., the values of $C_{11}^{(0)}$, $C_{12}^{(0)}$, and $C_{21}^{(0)}$. It follows from these equations that at $\epsilon = l/L \ll 1$ and $C_{11}^{(0)} \gg |C_{12}^{(0)}|$

$$C_{11}^{(0)} = \theta_{\max} - \theta_h = \frac{A}{4\epsilon}\left[1 \pm \left[1 - \frac{12\epsilon(1+A)}{A^2}\right]^{1/2}\right], \quad (53)$$

$$C_{21}^{(0)} = \eta(0) = \frac{4\epsilon}{3}(C_{11}^{(0)})^2 \,, \qquad C_{12}^{(0)} \equiv \theta_{\min} - \theta_h = -4\epsilon C_{11}^{(0)} \,. \quad (54)$$

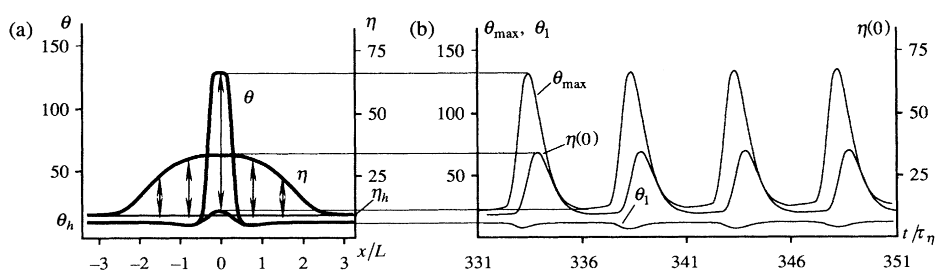

FIG. 3. (a) A pulsating spike auto-soliton; (b) the form of the autosoliton amplitude auto-oscillation, i.e., of the quantity $\theta_{\max} \cong C_{11}$, as well as of the quantities $\eta(0) = C_{21}$ and $\theta_1 = C_{12}$. Results of the numerical analysis of Eqs. (60)–(62) at $\epsilon = l/L = 10^{-2}$, $\alpha = 0.58$, and $A = 1$ are shown.

By substituting Eqs. (53) and (54) into Eqs. (44) and (45) we find the functions $\theta(x)$ and $\eta(x)$ describing the shape of a spike autosoliton [Fig. 1(b)]. We emphasize that, when $\epsilon \ll A^2 < 1$, these functions practically coincide with the results obtained for model (16),(17) on the basis of the theory of singular perturbations specially developed in Ref. [10] for narrow spike autosolitons [4,6].

One can see from Eqs. (44) and (53) that at $\epsilon \ll 1$ the amplitude of an autosoliton, i.e., the value of $\theta_{\max} - \theta_h = C_{11}^{(0)}$, decreases as the system's excitation level A decreases. At the point

$$A = A_b \cong \left[\frac{12l}{L} \right]^{1/2} , \tag{55}$$

where $d\theta_{\max}/dA = \infty$ and $d\eta(0)/dA = \infty$ [curves 3 in Figs. 1(c) and 1(d)], the autosoliton, which at $L \gg l$ has large amplitude

$$\theta_{\max} - \theta_h = \left[\frac{3L}{4l} \right]^{1/2} \gg 1 , \tag{56}$$

disappears abruptly.

Note that the autosoliton's threshold values determined by Eqs. (55) and (56) coincide with the quantities A_b and $\theta_{\max}$ from Eqs. (29) and (30) for a stratum if we replace the system's size $\mathcal{L}$ by the quantity $2L$. This quality determines the autosoliton localization region size where the relative change of the value of $\eta(x)$ is only of the order of one [Fig. 1(b)].

Note that if we replace $\cosh^{-2}(x/2l)$ in Eq. (44) with $\cosh^{-1}(x/l)$ we obtain the following expressions instead of Eqs. (53) and (54):

$$C_{11}^{(0)} = \frac{3\pi A}{32\epsilon} \left[1 \pm \left[1 - \frac{A_b^2}{A^2} \right]^2 \right] ,$$
$$A_b = \frac{32}{3\pi} \left[\frac{l}{L} \right]^{1/2} = \left[\frac{11.5l}{L} \right]^{1/2} , \tag{57}$$

$$C_{21}^{(0)} = \epsilon (C_{11}^{(0)})^2, \quad C_{12}^{(0)} = -\pi\epsilon C_{11}^{(0)} . \tag{58}$$

It is seen that Eqs. (57) and (58) do not differ greatly from Eqs. (53)–(55). The same conclusion can be made about the substitution of the function $\cosh^{-1}(x/L)$ instead of $\exp(-|x|/L)$ into Eqs. (44) and (45). In other words, the main parameters of autosolitons, as well as those of strata (Sec. IV), weakly depend on the details of the shape of $\theta(x)$ and $\eta(x)$ [Figs. 1(c) and 1(d)]. To a greater extent this conclusion can be applied to the functional dependences of the autosoliton's characteristic parameters on the quantities A, l, and L, i.e., on the system's parameters.

We can see from Eqs. (55) and (57) that $A_b \ll 1$ when $L \gg 12l$. This result confirms the conclusion, drawn earlier on the basis of qualitative analysis of a concrete system [4], that autosolitons in the form of a localized highly nonequilibrium regions, some kind of a ball lightning, can be excited in a system slightly deviated from its thermodynamic equilibrium.

VII. STABILITY OF STATIC AND SELF-OSCILLATIONS OF PULSATING SPIKE AUTOSOLITONS

As is seen from the analysis of stability of the autosoliton, its shape does not change in the process of spontaneous formation of a pulsating autosoliton, i.e., the shape of distributions $\theta(x)$ and $\eta(x)$ remains unchanged and coincides with that of the static autosoliton [4,6]. This fact is also confirmed by the numerical analysis of pulsating spike autosolitons. That is why we can use Eqs. (44) and (45) with time-dependent coefficients C_{ij} for the study of stability of a static autosoliton and auto-oscillations of a spike autosoliton.

Substituting Eqs. (44)–(47) into Eq. (12) and calculating the corresponding integrals we find

$$\tfrac{2}{3}\tau_\theta \dot{C}_{11} + \tau_\theta \dot{C}_{12} = -\frac{1}{2l}\frac{\partial \Phi_1}{\partial C_{11}} , \tag{59a}$$

$$4\epsilon\tau_\theta \dot{C}_{11} + \tau_\theta \dot{C}_{12} = -\frac{2}{L}\frac{\partial \Phi_1}{\partial C_{12}} , \tag{59b}$$

$$\tau_\eta \dot{C}_{21} = -\frac{2}{L}\frac{\partial \Phi_2}{\partial C_{21}} . \tag{59c}$$

Substitute Eqs. (50)–(52) into Eqs. (59). As a result after obvious rearrangement at $\epsilon = l/L \ll 1$ and $C_{12} \ll |C_{12}|$ we obtain dynamic equations describing the time dependence of the autosoliton's characteristic parameters:

$$\tfrac{2}{3}\alpha \dot{C}_{11} = -C_{11}\left[\frac{4}{5} - \frac{4A\theta_h}{3(C_{21}+\eta_h)} \right] + C_{12} + \frac{A}{3(C_{21}+\eta_h)}\left[\tfrac{8}{5}C_{11}^2 + 4C_{11}C_{12} + C_{12}^2 \right] , \tag{60}$$

$$\dot{C}_{21} = -2\left[C_{21} - \theta_h(C_{12} + 4\epsilon C_{11}) - \tfrac{1}{2}C_{12}^2 - \tfrac{4}{3}\epsilon C_{11}^2 \right] , \tag{61}$$

$$\alpha \dot{C}_{12} = \tfrac{4}{5}\epsilon C_{11} - C_{12}\left[1 - \frac{2A\theta_h}{C_{21}+\eta_h} \right] - \frac{A}{C_{21}+\eta_h}\left(2C_{21} - \tfrac{2}{3}C_{12}^2 + \tfrac{8}{15}\epsilon C_{11}^2 \right) , \tag{62}$$

where $\alpha = \tau_\theta/\tau_\eta$, and time is measured in units of τ_η.

To investigate stability of a static autosoliton we seek the solution of Eqs. (60)–(62) in the form of Eq. (35), where $C_{ij}^{(0)}$ are defined by Eqs. (53) and (54). Linearizing Eqs. (60)–(62) with respect to the small perturbations δC_{ij} and using $\epsilon \ll 1$ we obtain an equation which determines the value of γ in Eq. (35):

$$\lambda^3 - a\lambda^2 + b\lambda - c = 0 , \tag{63}$$

where

$$\lambda=\alpha\gamma,\quad a=2\alpha-0.2\left[1+\epsilon\left[9-\frac{12}{A}-\frac{36}{A^2}\right]\right],$$

$$b=4.4\alpha\left[1-\epsilon\left[3.5+\frac{0.5}{A}-\frac{3.3}{A^2}\right]\right]$$

$$-1.2\left[1-\epsilon\left[17+\frac{13}{A}+\frac{6}{A^2}\right]\right],$$

$$c=2.4\alpha\left[1-\epsilon\left[9-\frac{3.4}{A}-\frac{6}{A^2}\right]\right].$$

Analysis of this equation reveals that, when $\epsilon \ll 1$ and $C_{21}^{(0)}, C_{11}^{(0)} \gg |C_{12}^{(0)}|$, the smaller-amplitude autosoliton corresponding to the minus in Eq. (53) are unstable regardless of α. In other words, the lower branches of the curves $\theta_{\max}(A)$ and $\eta(0)(A)$ indicated in Figs. 1(c) and 1(d) by the dashed lines correspond to unstable states. These conclusions qualitatively coincide with those obtained in Sec. V for a stratum in a small-size system and confirm the results of the general theory of autosoliton stability [4,6].

It follows from analysis of Eq. (63) that the larger-amplitude static spike autosolitons, corresponding to the plus in Eqs. (53) and (54), become unstable ($\mathrm{Re}\gamma<0$) with respect to an increase of the perturbation given by Eq. (35) with $\mathrm{Im}\gamma \equiv \omega_0\tau_\eta \cong (6\alpha)^{1/2}$ if the condition, which can be written at $\epsilon \ll 1$ in the form

$$\alpha<0.59+\epsilon\left[1.33-\frac{3.03}{A}+\frac{0.15}{A^2}\right], \tag{64}$$

is satisfied.

From Eq. (64) we can draw conclusions qualitatively coinciding with those given in Sec. V where condition (38) of instability of a spike stratum in a small-size system was discussed. In particular, it follows from (64) that at $\epsilon \ll 1$ spike autosolitons are unstable when $\alpha<0.59$, practically regardless of the system's excitation level (the quantity A). This result confirms the conclusion of the general theory of autosoliton stability [4,6] according to which a static narrow spike autosoliton is unstable when the value of α is less than a quantity of order one. As a result of such instability a pulsating spike autosoliton may occur in the system.

Obtained on the basis of the presented variational method, Eqs. (60)–(62) describe nonlinear arbitrary amplitude auto-oscillations of the main parameters of a pulsating autosoliton. The results of numeric analysis of these equations show that the amplitude of auto-oscillations of the quantity $\theta_{\min}-\theta_h \equiv C_{12}$, as it should have been expected, is small in comparison with the amplitude of auto-oscillations of the quantities $\theta_{\max}-\theta_h \equiv C_{11}$ and $\eta(0) \equiv C_{21}$ (Fig. 3).

Auto-oscillations occur in a relatively narrow range of values of α, and at $A<1$ they are nearly periodic. For example, at $\epsilon=10^{-2}$ and $A=0.9$ stationary auto-oscillations, i.e., pulsating autosolitons (Fig. 3), occur only when $0.58<\alpha<0.6$. At $\alpha>0.6$ a static autosoliton, which in accordance with condition (64) is stable, can be exited in the system; at $\alpha<0.58$ all the initial perturbations are damped and the system relaxes into a homogeneous state. At $\epsilon=10^{-2}$ and $A=0.9$ a pulsating autosoliton occurs only when $0.56 \le \alpha \le 0.6$, and at $A=0.5$—only when $0.51 \le \alpha \le 0.53$.

A state in the form of a pulsating autosoliton has a very small region of attraction, i.e., it can be excited only when the initial values of C_{11}, C_{12}, and C_{21} are chosen close enough to their stationary values $C_{11}^{(0)}$, $C_{12}^{(0)}$, and $C_{21}^{(0)}$, which are determined by Eqs. (53) and (54). The amplitude of oscillations of the autosoliton decreases as A decreases, and at A slightly greater than the quantity A_b [Eq. (55)] the pulsating large-amplitude autosoliton disappears abruptly.

VIII. THE TYPE OF BIFURCATION: THE CONDITION OF FORMATION OF SMALL-AMPLITUDE PERIODIC STRUCTURES

The presented variational method allows us to fairly easily analyze the type of bifurcations at the points where the original state (a homogeneous, as well as inhomogeneous one) ceases to be stable. For simplicity let us illustrate this by an example of a homogeneous state of a system. Such a state, as pointed out in Sec. VI, becomes unstable at the point $A=A_c$ determined by condition (43). This instability takes place due to the increase of critical fluctuations with the period $\mathcal{L}_0=2\pi(lL)^{1/2} \gg l$ at $L \gg l$ [5,6]. In other words, $A=A_c$ at the point where the homogeneous state (42) of the considered model (16),(17) branches with the inhomogeneous small-amplitude harmonic solution in the form

$$\theta(x)=C_{11}\cos\left[\frac{2\pi x}{\mathcal{L}_0}\right]+C_{12},$$
$$\eta(x)=C_{22}\cos\left[\frac{2\pi x}{\mathcal{L}_0}\right]+C_{21}. \tag{65}$$

Substituting Eqs. (65) into Eqs. (19)–(22) and integrating we find the functionals Φ_1 and Φ_2:

$$\Phi_1\mathcal{L}^{-1}=\frac{1}{2}\left[\frac{\pi l}{\mathcal{L}}\right]^2 C_{11}^2-\tfrac{1}{2}C_{12}+\tfrac{1}{4}C_{12}^2+\tfrac{1}{8}C_{11}^2$$
$$-\frac{A}{6C_{22}}(C_{12}^3C_{22}\beta+3C_{12}^2C_{11}(1-C_{21}\beta)+\tfrac{1}{2}C_{11}^3$$
$$+\frac{C_{11}^3C_{21}^2}{C_{22}^2}(1-C_{21}\beta)-\frac{3C_{12}C_{11}^2C_{21}}{C_{22}}(1-C_{21}\beta)], \tag{66}$$

$$\Phi_2\mathcal{L}^{-1}=\frac{1}{2}\left[\frac{\pi L}{\mathcal{L}}\right]^2 C_{22}^2-\tfrac{1}{2}C_{21}C_{12}^2+\tfrac{1}{4}C_{21}^2+\tfrac{1}{8}C_{22}^2$$
$$-\tfrac{1}{4}C_{21}C_{11}^2-\tfrac{1}{2}C_{22}C_{12}C_{11}, \tag{67}$$

where $\beta=(C_{21}^2-C_{22}^2)^{-1/2}$. The substitution of Eqs. (66) and (67) into Eq. (8) yields a set of algebraic equations in the coefficients $C_{11}^{(0)}$, $C_{12}^{(0)}$, $C_{22}^{(0)}$, and $C_{21}^{(0)}$ determining parameters of periodic static states in the form (65) at the values of A close enough to A_c. It follows from this set of equations that

$$C_{21}^{(0)}=(C_{12}^{(0)})^2+\tfrac{1}{2}(C_{11}^{(0)})^2 ,$$
$$C_{22}^{(0)}=C_{12}^{(0)}C_{11}^{(0)}(2B_L)^{-1} , \tag{68}$$

$$C_{12}^{(0)}=1+\beta A(C_{12}^{(0)})^2 + A(1-\beta C_{21}^{(0)})\frac{C_{11}^{(0)}}{C_{22}^{(0)}}\left[2C_{12}^{(0)}-\frac{C_{11}^{(0)}C_{21}^{(0)}}{C_{22}^{(0)}}\right] , \tag{69}$$

$$C_{11}^{(0)}B_l=\frac{A}{4}\frac{(C_{11}^{(0)})^2}{C_{22}^{(0)}}+\frac{A}{6}\frac{1-\beta C_{21}^{(0)}}{C_{22}^{(0)}} \times\left[3(C_{12}^{(0)})^2+\frac{3(C_{11}^{(0)}C_{21}^{(0)})^2}{(C_{22}^{(0)})^2} - \frac{6C_{12}^{(0)}C_{11}^{(0)}C_{21}^{(0)}}{C_{22}^{(0)}}\right] , \tag{70}$$

where

$$B_L=\tfrac{1}{4}+\left[\frac{\pi L}{\mathcal{L}}\right]^2,\quad B_l=\tfrac{1}{4}+\left[\frac{\pi l}{\mathcal{L}}\right]^2 . \tag{71}$$

Dividing Eq. (70) by Eq. (69) and using Eq. (68) we obtain an equation in the amplitude of a harmonic, more exactly, in the quantity $a=C_{11}^{(0)}/C_{12}^{(0)}$. It follows from this equation with the accuracy of order $\alpha^4 \ll 1$ that

$$a^2=(AD)^{-1}\left[B_LB_l(1+A)-\tfrac{1}{2}A\left[\frac{\pi L}{\mathcal{L}}\right]^2\right] , \tag{72}$$

where

$$D=(B_L^4-\tfrac{1}{16})-\tfrac{3}{2}(B_L-\tfrac{1}{2})^2(B_L^2-\tfrac{1}{4})-\tfrac{3}{8}B_L^2+\tfrac{1}{2}B_L^4 -B_L^3+\tfrac{1}{2}B_L+\frac{1}{2}\left[\frac{\pi L}{\mathcal{L}}\right]^2\frac{B_l}{B_L} . \tag{73}$$

At the beginning of this section it was pointed out that at the bifurcation point $A=A_c$ the critical fluctuation period is $\mathcal{L}=\mathcal{L}_0\cong 2\pi(lL)^{1/2}$. Substituting this quantity into Eqs. (71)–(73) and taking into account that $\epsilon<1$ and, at A close to A_c, the quantity $C_{12}^{(0)}\cong\theta_h$ we find approximately

$$(C_{11}^{(0)})^2=\frac{32\theta_h^2\epsilon^3}{A}\left[\frac{A(1-2\epsilon-\epsilon^2)-(1+\epsilon)^2}{(18\epsilon^2+2\epsilon-3)}\right] =\frac{16\epsilon^3(A-A_c)\theta_h^2}{9A(\epsilon+\epsilon_1)(\epsilon-\epsilon_c)(\epsilon_m-\epsilon)(\epsilon+\epsilon_2)} , \tag{74}$$

where $\epsilon_1\cong 0.47$, $\epsilon_c\cong 0.36$, $\epsilon_m\cong 0.41$, $\epsilon_2\cong 2.41$.

Equation (74) gives $C_{11}^{(0)}\sim|A-A_c|^{1/2}$. This result conforms with Landau's general conclusion [1] that the amplitude of a quasiharmonic inhomogeneous state increases proportionally to the square root of the supercriticality, i.e., of the quantity $|A-A_c|$.

The quantity A_c in Eq. (74) is defined by Eq. (43) from which it follows that homogeneous state instability (and, consequently, the considered bifurcation of solutions) takes place only at $\epsilon<\epsilon_m$. Therefore, according to Eq. (74) the supercritical solution bifurcation, indicated by curve 7 in Figs. 1(c) and 1(d), is realized only if the condition

$$\epsilon_c<\epsilon<\epsilon_m \tag{75}$$

is satisfied. In this case a soft mode of formation of quasiharmonic states, the amplitude of which increases monotonically under the law $(A-A_c)^{1/2}$ as A increases, takes place. As $\epsilon_m=0.41$ and $\epsilon_c=0.36$, it follows from condition (75) that such a mode can be realized only at extremely rigorous requirements to the system's parameters and only at the values of A very close to the quantity A_c. In other words, a soft mode of spontaneous formation of small-amplitude quasiharmonic states, as stressed in Refs. [5,6], is very difficult to realize in numerical investigations, the more so in experiments.

It can be seen from Eq. (74) that when $\epsilon\to\epsilon_c$, the form of branching solutions tends to the vertical line [curve 6 in Figs. 1(c) and 1(d)]. The demarcation value of $\epsilon=\epsilon_c$ determines the boundary between the soft and hard modes of excitation of quasiharmonic inhomogeneous states.

At $\epsilon<\epsilon_c\cong 0.36$, according to Eq. (74), quasiharmonic small-amplitude states occur only at $A<A_c$ [curve 5 in Figs. 1(c) and 1(d)], i.e., a subcritical solution bifurcation takes place. In this case large-amplitude structures, including those in the form of periodically arranged strata with the shape like one of the stratum shown in Fig. 1(a), arise abruptly at the point $A=A_c$. As a result, pulsating and even stochastically oscillating quasiharmonic states can occur in the system.

Substituting Eqs. (65)–(67) into the general equation (12) it is easy to obtain the equations describing the dynamics of formation of small-amplitude states with an arbitrary period. We emphasize that the variational method developed for the study of small amplitude states allows a fairly easy analysis of the types of bifurcation occurring near points of instability of both homogeneous and inhomogeneous states, including, in the form of periodically arranged large-amplitude strata.

This method can be easily generalized to any other system. To do this it is necessary to expand functions in the equations describing the properties of such a system into a power series in the small amplitudes of the corresponding states (in particular, harmonic ones) and confine ourselves to the terms quadratic in the amplitudes. Specifically, it can be shown that results following from Eq. (74) remain qualitatively valid for other active distributed media as well.

IX. ON THE DERIVATION OF THE EQUATIONS DESCRIBING INTERACTION BETWEEN AUTOSOLITONS AND TURBULENCE IN ACTIVE DISTRIBUTED MEDIA

Turbulence in the form of stochastically oscillating inhomogeneous macroscopic states can be observed in aero- and hydrodynamic systems [11,14], nonequilibrium electron-hole and gas plasma, various chemical media, and other active distributed systems [5,6]. One of the properties of turbulence is the absence of space correlation in time-space stochastic oscillations. However, in the majority of approaches this essential property of turbulence is ignored, though it can be naturally taken into account within the framework of the presented variational method.

In many systems turbulence is a process of the randomly appearing and disappearing of autosolitons at various points of a system. This process is connected with nontrivial properties and the complex character of interaction between autosolitons [5,6]. First this interpretation of turbulence was developed in Refs. [15,16] with application to active distributed media and then used to explain the turbulence picture observed in flows of fluids at the near-critical Reynolds numbers [11].

In Sec. V it was found that auto-oscillations even of a solitary stratum, can be of a stochastic character (Fig. 2). Obviously, auto-oscillations of several pulsating strata or autosolitons placed at the distance $\mathcal{L}_k > L$ from one another can be uncorrelated with respect not only to time but to space as well, i.e., they can appear as turbulence of some kind.

Turbulence can also occur in systems at $\alpha > 1$, when pulsating strata and autosolitons do not take place (see Secs. V and VII). In this case turbulence is connected with the complicated dynamics of interaction between autosolitons, as well as with some of their specific properties [15,16,5,6].

Interaction between autosolitons and dynamics of their behavior can be studied on the basis of equations which can be derived from the general equations (11) or (12). To derive such equations one should, by analogy with Eqs. (44) and (45), write down the inhomogeneous state, corresponding to spike autosolitons placed at the distances $\mathcal{L}_k > L$ from one another, in the form

$$\theta(x) = \sum_k \left\{ C_{11}^{(k)} \cosh^{-2}\left[\frac{x-\mathcal{L}_k}{2l}\right] + C_{12}^{(k)} \exp\left[-\frac{|x-\mathcal{L}_k|}{L}\right] + C_{13}\right\}, \qquad (76)$$

$$\eta(x) = \sum_k \left\{ C_{21}^{(k)} \exp\left[-\frac{|x-\mathcal{L}_k|}{L}\right] + C_{22}\right\}, \qquad (77)$$

where $k = 0,1,2,\ldots,N$ is the number of autosolitons. The substitution of Eqs. (76) and (77) into Eqs. (19)–(22) yields expressions for the functionals Φ_1 and Φ_2, in which interaction between autosolitons is described by the terms proportional to the overlap integrals of functions $\theta(x)$ and $\eta(x)$ corresponding to different autosolitons [subscripts k in Eqs. (76) and (77)].

When the distance $\mathcal{L}_k$ between autosolitons essentially exceeds L, we can confine ourselves to considering only the interaction between the adjacent autosolitons. In other words, when calculating integrals in Eq. (11) or (12), we can take into account only the overlap of the functions describing the adjacent autosolitons. It allows an easy derivation of equations describing dynamics of several interacting spike autosolitons.

These equations can have solutions in the form of stochastically oscillating autosolitons which are not correlated in both time and space. The distances between the autosolitons, i.e., the quantities $\mathcal{L}_k$ in Eqs. (76) and (77), can also vary in the process of auto-oscillations.

In the stationary case we can find the value of A at which the solution, describing a state in the form of two autosolitons placed at a distance $\mathcal{L}_k$ from each other, disappears. In other words, we can find the dependence of the maximum distance between autosolitons, i.e., the quantity $\mathcal{L}_{\max}$, on the system's excitation level (parameter A). On the other hand, from these equations we can find the minimum distance $\mathcal{L}_{\min}$ at which the two autosolitons become unstable, to be more precise, find the dependence of $\mathcal{L}_{\min}$ on A. For certain systems there may exist a range of values of A where the condition $\mathcal{L}_{\min}(A) > \mathcal{L}_{\max}(A)$ is satisfied. In this case turbulence can occur even at $\alpha > 1$ [5,15]. Analysis also shows that at $\alpha > 1$ turbulence connected with the effect of self-annihilation of autosolitons in the process of their formation [5,6] occurs if the term $(\theta^2 - \eta)$ in Eq. (17) is multiplied by η.

X. CONCLUSION

It follows from the analysis carried out for the concrete system that the presented variational method allows us to study analytically the shape, evolution, and dynamics of rather complicated one-dimensional states with relatively high accuracy, without using numerical methods. Let us stress that, on the basis of the general equation (11) or (12), it is fairly easy to generalize the obtained results to two and three dimensions.

Note that to efficiently use the method presented it is necessary to find an approximate shape of inhomogeneous states occurring in a given system using numerical or qualitative analysis. For many systems such an analysis can be carried out on the basis of the general qualitative theory of dissipative structures in active distributed media [4,6].

The presented variational method can be useful for the study of inhomogeneous macroscopic states in many other dissipative systems, including those in aero- and hydrodynamic systems, in particular, of the states occurring at convective instability of a fluid layer (the Bénard problem). When taking such an approach one should first study an isolated Bénard cell (which in this case is an autosoliton) in more detail, and then study properties and interaction between such cells and autosolitons; further, on the basis of the ideas presented, one can derive ordinary differential equations describing hydrodynamic turbulence, to be more precise, characteristic parameters of the Bénard cells stochastically oscillating in time and space. We believe that this approach will be useful for deriving simplified equations describing turbulence in other hydrodynamic systems as well.

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Nauka, Moscow, 1976 and 1986), Vols. 5 and 6 [in Russian].

[2] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley Interscience, New York, 1977).

[3] B. D. Vujanovic and S. E. Jones, *Variational Methods in Nonconservative Phenomena* (Academic, London, 1989).

[4] B. S. Kerner and V. V. Osipov, Usp. Fiz. Nauk. **157**, 201 (1989) [Sov. Phys. Usp. **32**, 101 (1989)].

[5] B. S. Kerner and V. V. Osipov, Usp. Fiz. Nauk **160**, 1 (1990) [Sov. Phys. Usp. **33**, 679 (1990)].

[6] B. S. Kerner and V. V. Osipov, *Autosolitons* (Nauka, Moscow, 1991) [in Russian] (to be published in English, Kluwer, Dordrecht, 1993).

[7] H. Haken, *Advanced Synergetics* (Springer-Verlag, Berlin, 1983).

[8] V. V. Gafiichuk, I. A. Lubashevskii, and V. V. Osipov, *Dynamics of Surface Pattern Formation in Systems with Free Boundaries* (Naukova Dumka, Kiev, 1990) [in Russian].

[9] H. Meinhardt and A. Gierer, J. Cell. Sci. **15**, 321 (1974); J. Theor. Bior. Diol. **85**, 429 (1980).

[10] A. L. Dubitsky, B. S. Kerner, and V. V. Osipov, Dokl. Akad. Nauk **308**, 857 (1989) [Sov. Phys. Dokl. **34**, 906 (1989)].

[11] M. I. Rabinovitch and M. M. Suschyk, Usp. Fiz. Nauk **160**, 3 (1990) [Sov. Phys. Usp. **33**, 1 (1990)].

[12] V. S. Anishenko, *Complex Oscillations in Simple Systems* (Nauka, Moscow, 1990) [in Russian].

[13] I. A. Lubashevskii and V. V. Osipov, Physica D (to be published).

[14] Yu. L. Klimontovich, *Turbulent Motion and Structure of Chaos* (Kluwer, Dordrecht, 1991).

[15] B. S. Kerner and V. V. Osipov, Dokl. Akad. Nauk **270**, 1104 (1983) [Sov. Phys. Dokl. **28**, 485 (1983)].

[16] B. S. Kerner and V. V. Osipov, Mikroelektronika **14**, 389 (1985) [in Russian].